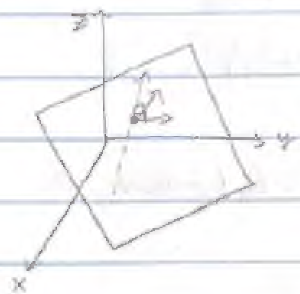
09/01/21

12.4 Cross Product

*NOTE: Everything today lives in $\mathbb{R}^3$

Goal: Given two vectors, construct a third (nonzero if possible) vector, orthogonal to both



Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

Suppose $\vec{\omega} = \langle \omega_1, \omega_2, \omega_3 \rangle$ is the desired vector, what must $\vec{\omega}$ satisfy?

Because $\vec{\omega}$ is orthogonal to both $\vec{u}$ and $\vec{v}$:

$$\begin{cases} 0 = \vec{\omega} \cdot \vec{u} = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 & (1) \\ 0 = \vec{\omega} \cdot \vec{v} = \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3 & (2) \end{cases}$$

Multiply (1) by v_3 and (2) by u_3 to obtain:

$$\begin{cases} 0 = v_3(\vec{\omega} \cdot \vec{u}) = \omega_1(u_1 v_3) + \omega_2(u_2 v_3) + \omega_3(u_3 v_3) & \text{Eq. } 1^* \\ 0 = u_3(\vec{\omega} \cdot \vec{v}) = \omega_1(v_1 u_3) + \omega_2(v_2 u_3) + \omega_3(u_3 v_3) & \text{Eq. } 2^* \end{cases}$$

Aside: $-ax + by = 0$ has a solution $x = b$, $y = a$ $\Rightarrow -ab + ba = 0$

Subtracting 2^* from 1^*:

$$0 = v_3(\vec{\omega} \cdot \vec{u}) - u_3(\vec{\omega} \cdot \vec{v})$$

$$= \omega_1(u_1 v_3 - u_3 v_1) + \omega_2(u_2 v_3 - u_3 v_2)$$

$$= -\omega_1(-(u_1 v_3 - u_3 v_1)) + \omega_2(u_2 v_3 - u_3 v_2)$$

Hence we have a solution:

$$\begin{cases} \omega_1 = u_2 v_3 - u_3 v_2 \\ \omega_2 = -(u_1 v_3 - u_3 v_1) \end{cases}$$

Now we plug back into ① to obtain w_3.

$$0 = w_1u_1 + w_2u_2 + w_3u_3$$
$$= (u_2v_3 - u_3v_2)u_1 + (-(u_1v_3 - u_3v_1))u_2 + w_3u_3$$
$$= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + w_3u_3$$
$$= (u_2v_1 - u_1v_2)u_3 + w_3u_3$$
$$= u_3(u_2v_1 - u_1v_2 + w_3)$$

So, either $u_3 = 0$ or $w_3 = u_1v_2 - u_2v_1$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$
$$= \langle u_2v_3 - u_3v_2,\ -(u_1v_3 - u_3v_1),\ u_1v_2 - u_2v_1 \rangle$$

Now we can check this to verify that $\vec{w}$ satisfies:

$$\begin{cases} \vec{w} \cdot \vec{u} = 0 \\ \vec{w} \cdot \vec{v} = 0 \end{cases}$$

Determinants

DEFINITION: The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

The determinant of a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

is: $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$= a(ek - fh) - b(dk - fg) + c(dh - eg)$$

Ex. Find the determinant of $\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det \begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - (-2) \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= 1\left(-1(1) - 0(1)\right) - (-2)\left(-1(1) - 0(1)\right) + 3\left(-1(0) - (-1)(0)\right)$$

$$= 1(-1) + 2(-1) + 3(0)$$

$$= -1 - 2 + 0 = \boxed{-3}$$

Turns Out: That vector is a symbolic determinant

$$\begin{array}{c} \\ \vec{u} \rightarrow \\ \vec{v} \rightarrow \end{array} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \hat{\imath}\left(u_2 v_3 - u_3 v_2\right) - \hat{\jmath}\left(u_1 v_3 - u_3 v_1\right) + \hat{k}\left(u_1 v_2 - u_2 v_1\right)$$

$$= \left\langle u_2 v_3 - u_3 v_2,\ -(u_1 v_3 - u_3 v_1),\ u_1 v_2 - u_2 v_1 \right\rangle$$

This is the same $\vec{w}$ we computed before!

<u>DEFINITION</u>: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$.

The cross product of $\vec{u}$ with $\vec{v}$ is

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

<u>NOTE</u>: The cross product as an operation takes <u>two</u> <u>vectors</u> in $\mathbb{R}^3$ and creats another vector in $\mathbb{R}^3$

<u>Properties of Cross Product</u> (Algebraic)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

Also geometric $\rightarrow$

5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

<u>Properties of Cross Product</u> (Geometric)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
2. $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin(\theta)$ (θ is angle between $\vec{u}$ & $\vec{v}$)
3. $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u} \times \vec{v} = 0$